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A Benchmark Comparison of Numerical Methods for Infinite Prandtl Number Thermal Convection in Two-Dimensional Cartesian Geometry

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2. Definition of the Benchmark Problems

Four benchmark problems were chosen, progressing from a simple steady-state case to one in which the convection is intrinsically time-dependent. The Rayleigh numbers for all the benchmark problems are between 3×10^4 and 1.3×10^5, roughly 30 to 100 times the critical Rayleigh number. The first two cases treat steady convection in a simple geometry, a square box, under two heating modes – bottom heating and internal heating. The third case introduces a transient, time-dependent phase caused by a sudden increase in the Rayleigh number. The flow evolves from the initial state to a new steady state. This benchmark is useful for determining how the different techniques follow transient, time-dependent behavior. The fourth case is a calculation of persistent time-dependent convection in an elongated rectangular box, with bottom heating. For the steady-state cases, the bases of comparison are temperature and velocity fields. The first three calculations were done using a 32×32 spatial grid, and these cases were also computed with a 128×128 grid, providing a more accurate solution for comparison. For the time-dependent cases, the histories of the kinetic energy and heat transport (Nusselt number) are also computed.

All problems assume a Boussinesq fluid with uniform material properties. In dimensionless form, the equations to be solved are

$$-\nabla P = Ra\ T\ \hat{\mathbf{z}} + \nabla^2 \mathbf{u} \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \nabla^2 T + A, \tag{3}$$

in which $\mathbf{u}$ is the fluid velocity, T is the temperature, P is (reduced) pressure, A is the volumetric heat production, $\hat{\mathbf{z}}$ is the unit vertical vector in an

(x, z) coordinate system, and Ra is the Rayleigh number. These equations are made dimensionless using the layer depth d and the thermal diffusion time d^2/K, where K is the thermal diffusivity, as length and time scales.

For problems B1 and B4, the temperature difference ΔT between bottom and top surfaces is fixed and serves as the temperature scale. In these cases $A = 0$ and the Rayleigh number is defined as

$$Ra = \frac{\rho\alpha\Delta T d^3}{K\eta}, \tag{4}$$

in which ρ is the reference fluid density, α is the thermal expansion coefficient, g is the acceleration of gravity, and η is the dynamic viscosity.

In problems B2 and B3, volumetric heat production is prescribed such that

$$A = 1. \tag{5}$$

In these cases, the temperature scale is d^2H/k where H is the (uniform) heat source density and k is thermal conductivity. The Rayleigh number is the defined as

$$Ra = \frac{\rho\alpha g d^5 H}{kK\eta}. \tag{6}$$

Three global functionals are used for comparison of results – the average kinetic energy, the Nusselt number and average basal temperature. The average kinetic energy in a rectangle with unit height and length L is

$$U = \frac{1}{2L}\int_0^1\int_0^L (u^2 + w^2)dxdz, \tag{7}$$

and the Nusselt number at height z is defined as

$$Nu(z) = \frac{1}{L}\int_0^L \left(wT - \frac{\partial T}{\partial z} \right) dx, \tag{8}$$

where (u, w) are velocity components along (x, z). For problems with a basal heat flux boundary condition, an important functional is the average basal temperature

$$\bar{T}(0) = \frac{1}{L}\int_0^L T(x, 0)dx. \tag{9}$$

Precise descriptions of the benchmark problems are as follows.

B1. Steady-State Convection with Base Heating

This problem corresponds to Rayleigh-Bénard convection in a two-dimensional square box with stress-free isothermal boundaries. The parameter values are

$$A = 0, \qquad Ra = 77,927, \tag{10}$$

which is 100 times the critical Rayleigh number value for this geometry. Boundary conditions are

$$z = 0 \qquad w = \frac{\partial u}{\partial z} = 0; \quad T = 1; \tag{11}$$

$$z = 1 \qquad w = \frac{\partial u}{\partial z} = T = 0; \tag{12}$$

$$x = 0 \qquad u = \frac{\partial w}{\partial x} = \frac{\partial T}{\partial x} = 0; \tag{13}$$

$$x = 1 \qquad u = \frac{\partial w}{\partial x} = \frac{\partial T}{\partial x} = 0; \tag{14}$$

The initial conditions are arbitrary, as the final solution is a steady-state single cell.

B2. Steady-State Convection with Internal Heating

Here the geometry is identical to problem B1, but the energy source is internal heat production. The parameter values are

$$A = 1, \qquad Ra = 38,880, \tag{15}$$

(30 times the critical value), and the boundary conditions are

$$z = 0: \qquad w = \frac{\partial u}{\partial z} = \frac{\partial T}{\partial z} = 0; \tag{16}$$

$$z = 1: \qquad w = \frac{\partial u}{\partial z} = T = 0; \tag{17}$$

$$x = 0: \qquad u = \frac{\partial w}{\partial x} = \frac{\partial T}{\partial x} = 0; \tag{18}$$

$$x = 1: \qquad u = \frac{\partial w}{\partial x} = \frac{\partial T}{\partial x} = 0; \tag{19}$$

Again, the initial conditions are arbitrary as the final solution is a single, steady cell.

B3. Transient Adjustment from an Initial to a Final Steady State

In this problem, the initial state is the final state in calculation B2. At time t=0, the Rayleigh number is suddenly increased to 100 times the critical value, and a new steady state is reached after a transient adjustment. The boundary conditions are given by (16) and the parameter values are

$$A = 1, \qquad Ra = 129,600; \qquad t > 0. \tag{20}$$

The transient behavior is described by time series of the kinetic energy, the surface heat flux (Nusselt number at z=1) and the average basal temperature.

B4. Time Dependent Convection

This is a problem mof base-heated Rayleigh-Bénard convection in an elongated box, in which the initial and final states are single elongated rolls. The final state is time-dependent, with recurring boundary layer instabilities. The instabilities produce oscillations in kinetic energy and Nusselt number about mean (time-averaged) values, but the oscillations do not destroy the large-scale flow. Between the initial and final state the solution is a two-cell pattern for a prolonged period of time. The parameter values are

$$A = 0, \qquad Ra = 100,000, \tag{21}$$

with boundary conditions

$$z = 0 \qquad w = \frac{\partial u}{\partial z} = 0; \quad T = 1; \tag{22}$$

$$z = 1 \qquad w = \frac{\partial u}{\partial z} = T = 0; \tag{23}$$

$$x = 0 \qquad u = \frac{\partial w}{\partial x} = \frac{\partial T}{\partial x} = 0; \tag{24}$$

$$x = 2 \qquad u = \frac{\partial w}{\partial x} = \frac{\partial T}{\partial x} = 0; \tag{25}$$

The initial temperature distribution corresponds to a single-cell pattern.

$$T(x, z) = (1 - z) + 0.01 \sin(\pi z) \cos(\frac{1}{2}\pi x), \quad t = 0. \tag{26}$$